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Osculating Circles of Conics in Cayley-Klein Planes

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ABSTRACT

In the Euclidean plane there are several well-known methods of constructing an osculating (Euclidean) circle to a conic. We show that at least one of these methods can be "translated" into a construction scheme of finding the osculating non-Euclidean circle to a given conic in a hyperbolic or elliptic plane. As an example we will deal with the Klein-model of these non-Euclidean planes, as the projective geometric point of view is common to the Euclidean as well as to the non-Euclidean cases.

Key words: Cayley-Klein plane, elation, pencil of conics, osculating circle, curvature centre

MSC 2010: 51N30, 51M15, 53A35

Oskulacijske kružnice konika u Cayley-Klein-ovim ravninama

SAŽETAK

U euklidskoj ravnini postoji nekoliko dobro poznatih metoda konstrukcija oskulacijske kružnice konike. Cilj je te konstrukcije "translatirati" u neke od neeuklidskih ravnina. U članku se daje opća konstrukcija oskulacijske kružnice konike zadane s pet elemenata u euklidskoj ravnini. Pokazuje se da je konstruktivna metoda primjenjiva u hiperboličkoj i eliptičkoj ravnini. Budući da je projektivno geometrijsko gledište zajedničko euklidskom i neeuklidskim slučajevima, analogne se konstrukcije koriste na Klein-ovim modelima neeuklidskih ravnina.

Ključne riječi: Cayley-Klein-ova ravnina, elacija, pramen konika, oskulacijska kružnica, središte zakrivljenosti

1 Preliminary Remark

Although the problem of constructing an osculating circle at a point of a conic seems to be anachronistic in times of numerical approximation tools, knowledge about exact constructive methods is not at all obsolete, particularly since these methods are uniformly applicable. Beyond that, with the following projective geometric constructions of osculating circles, we place particular emphasis on synthetic argumentation, which is typically for geometry. Unfortunately Projective Geometry and Non-Euclidean Geometry in the sense of F. Klein does not have much space in nowadays Mathematics education such that valuable Geometry culture is in danger of vanishing. Our article might perhaps help to counteract these facts. The paper is also to be posed into the series of articles of classical Projective and Non-Euclidean Geometry initiated by the second author, see [5] - [8].

It is hard to say, how "well-known" the presented considerations are; they could be for example exercise material to lectures on classical Projective Geometry and not considered to be valuable enough to be published. To our knowledge lectures with related content still exist in Vienna (H. Stachel, H. Havlicek) and Graz (J. Wallner, O. Röschel), where they still belong to the syllabus in teachers education in Descriptive Geometry.

2 Euclidean Osculating Circles of Conics

We start with "permissible standard givens" of a conic in the Euclidean plane, i.e. from pair of conjugate diameters $\overline{AC}$, $\overline{BD}$ of an ellipse, from a pair of line elements (A,t_A), (B,t_B) of a parabola and from a pair of asymptotes (r,s) and a point A of a hyperbola. The problem is to find the osculating circle at the given point A.

In geometry courses for engineers one usually presents the construction recipe for the hyperosculating circles at the vertex of the conic; in lectures on differential geometry for Mathematicians this recipe is also presented and the analytic equality of 4^{th}order is proved by calculation in each of the three cases. But for all three cases there is a uniform projective geometric idea for the solution of this (more general) problem. This idea uses properties of osculating resp. hyperosculating pencils of conics. This unified explanation of elementary construction of hyperosculating circles at vertices of a conic might be not new, but it is, in our opinion, not at all so well-known as it is worthy to note.

In addition, the construction principle deduced from it can be used for all Cayley-Klein geometries, as it is shown in the following chapter only for the hyperbolic and the quasi-hyperbolic geometry (which is the dual geometry to the pseudo-Euclidean geometry) as an example.

Specifically, the construction follows two steps:

Step 1: Transform the given conic c by the standard shear transformation $\sigma_1 : c \to c'$ into another conic c', which osculates c at the point A, and has A as a vertex. The axis of the shear therefore is the tangent t_A in A. As all elations with the center N on t_A, this shear σ_1 maps the given conic c into c' which osculates the conic c at A. The osculating circle of one is also osculating circle for the other one.

Step 2: Construct the hyperosculating circle k_A of the conic c' at $A = A'$. For this one can use an additional elation σ_2 which also has t_A as its axis, but the center A and it should map c' into a hyperosculating conic c''. By demanding $c' \mapsto k_A$ the transformation σ_2 is uniquely defined; σ_2 transforms the point B' of the osculating conic c' (in the case of an ellipse this is a "neighbouring" vertex, in case of a hyperbola one of the asymptote's ideal point "at infinity", in the case of a parabola it is the additionally given point) into the point B'' of k_A. The normal from the (fixed) point $T = T' = T'' \in t_A$ ($T = t_A \cap t_B = t_{B'} \cap t_B$) to the chord $AB'' = AB'$ of circle k_A therewith passes through the center M_A of k_A.

The following figures (Figures 1-3) show the construction costs, which in each of the three cases needs only a few lines.

Elliptic case (Figure 1)

The point A is transformed into the vertex of an ellipse c' by the shear $\sigma_1 : c \to c'$ (tangent t_A at A is the axis of σ_1). Furthermore, $\sigma_2 : c' \to k_A$ with the same axis, but with center A, transforms a "neighbouring" vertex B' of c' into the point B'' of k_A. The line through the fixed point $T = T' = T''$ on t_A perpendicular to the chord $AB'' = AB'$ of the osculating circle k_A intersects the normal n of the conic c given at the point A in the center M_A of k_A.

Parabolic case (Figure 2)

The shear $\sigma_1 : c \to c'$ (axis is the tangent t_A at A) transforms the point A into the vertex of a parabola c', which osculates c. The midpoint H of the chord $\overline{AB}$, together with the point $T := t_A \cap t_B$, defines the diameter direction of the conic c, and therefore is mapped into the point H' on the line normal to t_A at T. For $\sigma_2 : c' \to k_A$ (the center is A) the line AH' is a fixed line; it contains the point $B'' = \sigma_2(B')$ of the hyperosculating conic k_A. The line through the fixed point $T = T' = T''$ perpendicular to the chord $AB'' = AH'$ of k_A passes through the sought-after curvature center M_A.

Hyperbolic case (Figure 3)

At first we construct the tangent t_A at the point A, (A is the midpoint of the tangent segment between the asymptotes). By a suitable shear σ_1 one can transform the point A into the vertex of the hyperbola c' whose asymptotes are r' and s' and the center is M'. The elation $\sigma_2 : c' \to k_A$ with center A transforms the ideal point $R' = B'$ of the asymptote r' of the conic c' into the point B'' of the hyperosculating conic k_A. Therefore, the line through the fixed point T on the tangent t_A perpendicular to $AB'' = AB'$ passes through the common curvature center M_A of the conics k_A, c' and c. As AB' is parallel to r', one only needs to draw the perpendicular line to the asymptote $r' = M'T$ at the point T.

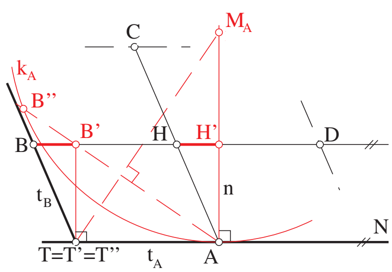

Figure1: Construction of the center M_A of the osculating circle k_A at the endpoint A of one of the given conjugate diameters of the ellipse c.

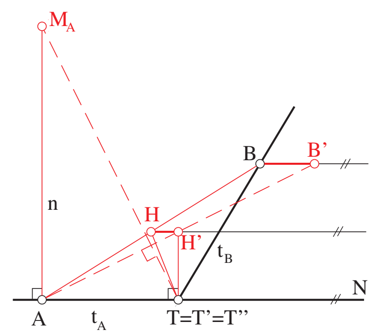

Figure 2: Construction of the center M_A of the osculating circle k_A at the point A of the parabola c given by two line elements (A, t_A), (B, t_B).

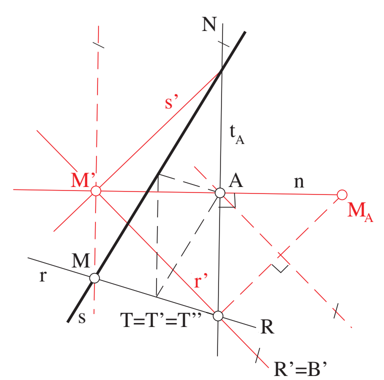

Figure 3: Construction of the curvature center M_A at the point A of the hyperbola $c$given by the pair of asymptotes (r, s) and the point A.

Remark 1 *Of course, there are other nice constructions of the Euclidean curvature circle at the point of a conic, too. We want to mention the one found in [1], which is based on the differential geometric investigations and goes as follows:*

"Choose two points P,Q on tangent t_A symmetric to A, draw the tangents from P and Q to the given conic c, receive the contact points P' and Q'. The normals p and q to the cords AP' and AQ' intersect the normal n in A in the points M_P and M_Q. Then M_A is the midpoint of the segment (M_P, M_Q). Also this construction could be transformed to CK-planes."

3 Curvature Centres of Conics which are Given by General Data in CK-planes

In the preceding chapter we started with affine-special given data of a conic. Now we choose a projective-geometric system of data defining a conic:

Let a conic c be given by two line elements (A,a), (B,b) and a point C in a (real) projective plane π equipped with an "absolute (regular or degenerated) polarity" $\pi^\perp$. (The plane π together with $\pi^\perp$ is therewith a Cayley-Klein plane, in short a CK-plane.)

These specifications of the conic c can easily be derived from all other given data, which define conic uniquely, by applying the theorems of Pappus-Pascal and Brianchon, see [2]. These givens are also appropriate for the analytic treatment of the conic, as they can be interpreted as a projective coordinate system.

Via $\pi^\perp$ the place of action is a projective plane with an orthogonality structure "$\perp$" and a concept of circles in the sense of Cayley-Klein; it is therefore a "projectively extended non-Euclidean plane", a CK-plane, see e.g. [3]. Among these CK-planes we want to exclude the so-called isotropic planes (see e.g. [4]) from further considerations, because their orthogonality structure $\perp$ is too degenerated. These CK-planes are treated separately in [7]. It turns out that for degenerated absolute polarity $\pi^\perp$ one could "swap" the two steps described in chapter 2: At first one constructs the hyperosculating parabola c' at the point A of the given conic c in the isotropic plane π with absolute point I at the ideal line u. This can be done with an elation with center A and axis a. Next, the parabola c' is transformed into the isotropic circle k with ideal point I using the shear with axis a.

In the last chapter, it will be shown that in some cases of CK-planes with degenerate absolute polarity one can easily find simplier constructions using one single elation alone.

The problem is to construct the curvature center M_A at the point A of the given conic c. (For the sake of simplicity we visualise again the Euclidean case in Figure 4, i.e. "$\perp$" and "circles" allow an elementary geometric interpretation.)

Step 1: determine the normal n of the conic c at the point A with respect to $\perp$; i.e. one needs to construct the "absolute conjugate line" n to the given tangent a of the conic c at A with respect to $\pi^\perp$. The absolute pole N of the normal n is a point of a and it has to be used as the center of an elation $\sigma_1 : c \to c'$ with axis a. One still needs a related pair of points to define elation σ_1 uniquely.

Step 2: the construction of a conic point D on the (fixed) collineation ray on BN using the theorem of Pappus-Pascal. (Naturally, it would be the same, if we constructed the point $E \in c$ on the collineation ray CN.) In Figure 4 the necessary lines are shown:

$$D := BN \cap AP, \quad \text{whereby,} \quad P := BC \cap \big((a \cap b) \vee (BN \cap AC)\big).$$

Step 3: determination of the 4^{th} harmonic point H to N with respect to the pair (B,D) on line BN. This point H shall be related to $H' := n \cap BN$ in σ_1 and now $\sigma_1 : c \to c'$ is uniquely determined.

Step 4: corresponds to step 2 in chapter 2 to get $\sigma_2 : c' \to k_A$. Explicitly, one only needs the center A of elation σ_2 and the fact that the point $B'' := \sigma_2(B')$ on k_A belongs to the chord $AB'' = AB'$ of k_A, which is fixed under σ_2 where $B' := \sigma_1(B)$. Therewith the wanted curvature center M_A is the intersection point of the normal n with the line s, which is the absolute-conjugate line to AB' through $a \cap b$.

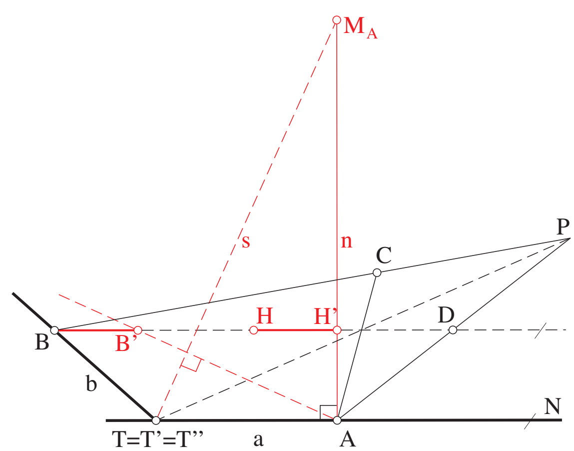

Figure 4: Construction of the curvature center M_A in a point A of a conic c, which is given by two line elements (A,a), (B,b) and an additional point C in the (projective extended) elementary geometric plane. (Explanation of the construction see above.)

4 *h*-Curvature Circles of a Conic in the Projective Model of a Hyperbolic Plane

As an example for the construction of the osculating circle in a non-Euclidean plane with regular absolute polarity $\pi^{\perp}$ the complete solution will be given in a hyperbolic plane, see Figure 5.

According to the projective geometric background of the idea of the construction, it is somehow natural to use the classical projective model of such a h-plane, (c.f. also [2]). This means that the place of action is essentially the inner domain of a (real) "absolute conic" u, which can be taken as an ordinary circle in elementary geometric sense. Given a conic c, the problem is to construct the h-curvature center M_A to the arbitrarily given point A (with tangent a).

It is expedient and practical to use a perspective collineation $\kappa_1 : u \to c$ to construct the conic c as collinearly related image to the absolute conic u. (In Figure 5 collineation κ_1 is defined with the center S and the axis s, and the related pair of points (A_1, A).) Note, that if s is absolute polar of S, the obtained conic c would be a circle.

Step 1: The osculating h-circle k of c at A has its center M on the h-normal n to a through A. So as a first step one needs to construct this n.

Step 2: Make A to a vertex of the conic c', which osculates c at A. For this we use a "projective shear", i.e. an elation σ_1 with axis a and centre $N \in a$, which is the absolute pole of n.

Step 3: Construct the hyperosculating circle of c' according to the description to Figure 4. In Figure 5 we used the points Q and $Q' =: \sigma_1(Q)$ to get c' from c and the special point B' and its tangent b. Finally we connect A with B' and erect the h-normal line to AB' through $T := a \cap b$, it intersects n in the h-curvature center M.

Step 4: If we do not use a graphics software like "Cinderella", where we can directly draw h-circles in a h-plane, we still have to construct the h-osculating circle k. This again can be done using a perspective collineation $\kappa_2 : u \to k$; it has the center M and the axis $m := \pi^{\perp}(M)$ and the related pair of points $(A \in c, A_2 \in u)$. Because of

$$u \xrightarrow{\kappa_1} c \xrightarrow{\sigma_1} c' \xrightarrow{\sigma_2} k \xrightarrow{\kappa_2^{-1}} u$$

The product of these perspective collineations must act as a projectivity β on u. So the mapping $\beta : c \to u \to k$ is determined by three pairs of points on u, among them A_1, A_2 and Q_1, Q_2.

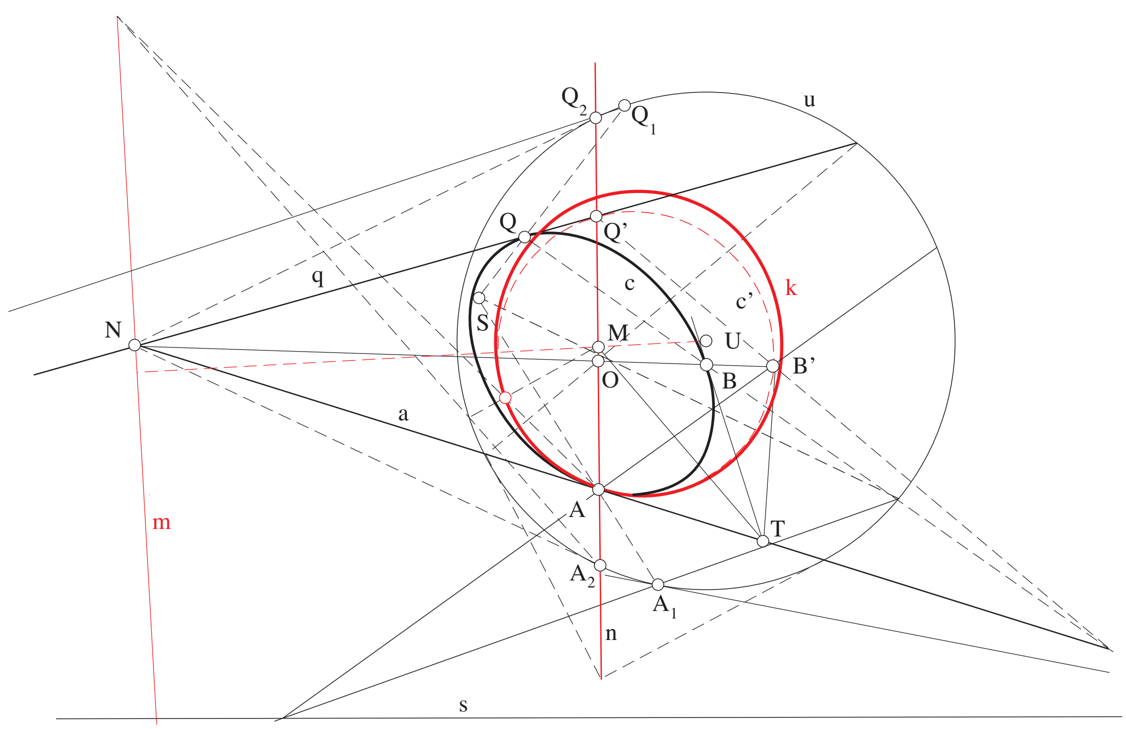

Figure 5: Osculating h-circle at the point A of the conic c. (Construction in the classical projective model of a h-plane.)

5 Curvature Circles of a Conic in the CK-Planes with Singular Absolute Polarity

To start with, we give an overview over those CK-planes, c.f. [3]:

$\pi^{\perp}$	acting in	"Absolute figure"	Name
elliptic involutoric projectivity	(ideal) line u	pair of imaginary points $I,J \in u$	Euclidean plane, $e-plane$
hyperb. involutoric projectivity	(ideal) line u	pair of real points $I,J \in u$	pseudo-Euclid. plane, $pe-plane$
elliptic involutoric projectivity	a pencil of lines u through (ideal) point U	pair of imaginary lines i,j through U	dual Euclid. plane, quasi-elliptic plane $qe-plane$
hyperb. involutoric projectivity	a pencil of lines u through (ideal) point U	pair of real lines i,j through U	dual pseudo-Euclid. plane, quasi-hyperbolic plane, $qh-plane$
degenerate inv. projectivity	(ideal) line u	point U and line u (self-dual figure)	isotropic plane, Galilean plane, $i-plane$

We treated the Euclidean case in Chapter 2 aiming at a unifying interpretation of the classical and well-known elementary constructions. We will now present constructions of osculating circles by using one single elation alone. Let us begin with

1) *Pseudo-Euclidean case*

Figure 6 shows the projective model of a pe-plane and the construction of a pe-circle osculating a conic c at the point A.

Explanation to Figure 6:
The conic c and its line element (A,a) in the pe-plane with absolute points $I,J \in u$ are given. Now we choose A as the center of an elation κ and construct its axis z: (Also this type of elations is osculation preserving!) With center A project I,J onto c getting I', J'. Intersect u with the line $u' = I'J'$, get a fixed point F and $AF =: z$. Now κ is well-defined and $\kappa(c) =: k$ is the desired osculating pe-circle.

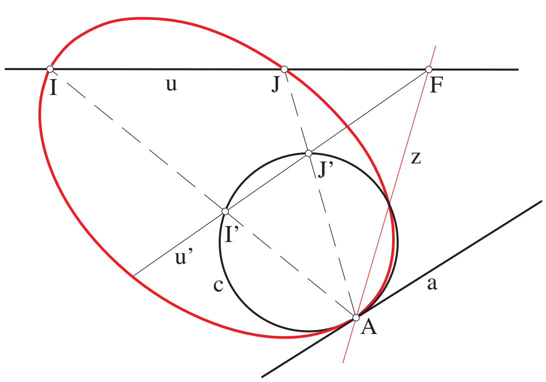

Figure 6: Osculating pe-circle at the point A of the conic c. (Construction in a projective model of a pe-plane.)

Remark 2 *The same construction principle can be performed also in the Euclidean case. The imaginary rays AI and AJ are defined by the orthogonal-involution in the pencil with vertex A and this orthogonal involution induces in c an elliptic involutoric projectivity ρ with involution center R. Now we had to construct the polar line r to R with respect to c; r connects the imaginary points I' and J' and therefore is parallel to the elation axis z.*

2) *Quasi-hyperbolic case*

Without loss of generality, let the absolute figure of the quasi-hyperbolic plane be a pair of parallel lines i,j. The conic c and its line element (A,a) are given. A qh-circle k is a conic k touching both absolute lines i and j. The construction is now dual to the one of Figure 6, see Figure 7.

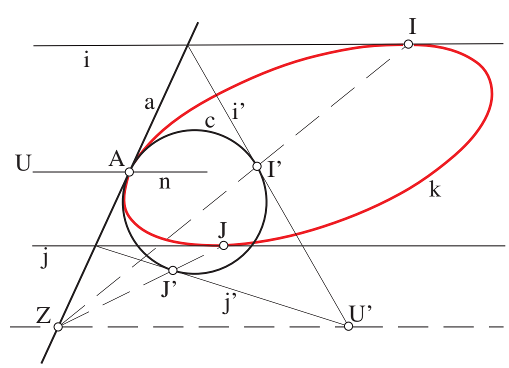

Figure 7: Osculating qh-circle at the point A of the conic c. (Construction in a qh - plane with an absolute figure $\{U;i,j\}$.)

Explanation to Figure 7:
Intersect the absolute lines i and j with the tangent a of the conic c at A. The tangents i', j' from these intersection points to c intersect at the point U' which corresponds to the absolute point U in the desired elation κ with axis a. Line UU' intersects a at the center Z of κ such that κ is well-defined by $\{Z, a, (U, U')\}$ and $k := \kappa(c)$ is the desired osculating qh-circle. Note that UA represents the qh-normal n of the c at A.

Remark 3 *The same construction principle can be performed also in the quasi-elliptic case. The construction "dualises" that of the Euclidean case. In the isotropic case, because of self-duality, one can use both, the principle of the "qh-construction" as well as that of the "pe-construction" to find the needed elation. As the construction is obvious, it can be left to the reader to practice.*

References

[1] G. Beyer, Untersuchungen zur Krümmung von Kurven zweiter Ordnung in der Ebene und Kurven vierter Ordnung erster Art im Raum. *Beiträge zur Algebra und Geometrie* **11** (1981), 65–73.

[2] H. Brauner, *Geometrie projektiver Räume 2*, BI. Mannheim-Wien-Zürich (1976)

[3] O. Giering, *Vorlesungen über höhere Geometrie*, Vieweg, Braunschweig (1982)

[4] H. Sachs, *Ebene isotrope Geometrie*, Vieweg, Braunschweig-Wiesbaden, (1987)

[5] A. Sliepčević, Iz prizemlja više geometrije. *KoG* **5** (2001), 73.

[6] A. Sliepčević, Eine Anwendung der perspektive Kollineation. *KoG* **5** (2001), 51–55.

[7] A. Sliepčević, M. Katić-Žlepalo, Osculating circles of conics in the isotropic plane. *Slovensky časopis pre geometriu a grafiku* **5** (2008), No.10, 21–26.

[8] A. Sliepčević, N. Kovačević, Hyperosculating circles of the conics in the Pseudo-Euclidean plane. (to be submitted).

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